

Semiclassical four-point functions in $AdS_5 \times S^5$

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Abstract

We consider a semiclassical (large string tension $\sim \sqrt{\lambda}$) limit of 4-point correlator of two “heavy” vertex operators with large quantum numbers and two “light” operators. It can be written in a factorized form as a product of two 3-point functions, each given by the integrated “light” vertex operator on the classical string solution determined by the “heavy” operators. We check consistency of this factorization in the case of a correlator with two dilatons as “light” operators. We study in detail the example when all 4 operators are chiral primary scalars, two of which carry large charge J of order of string tension. In the large J limit this correlator is nearly extremal. Its semiclassical expression is, indeed, found to be consistent with the general protected form expected for an extremal correlator. We demonstrate explicitly that our semiclassical result matches the large J limit of the known free $\mathcal{N} = 4$ SYM correlator for 4 chiral primary operators with charges $J, -J, 2, -2$; we also compare it with an existing supergravity expression. As an example of a 4-point function with two non-BPS “heavy” operators, we consider the case when the latter are representing folded spinning with large AdS spin and two “light” states being chiral primary scalars.

Dedicated to the memory of V.Ya. Fainberg

1 Introduction

A correlator of operators in $AdS_5 \times S^5$ string theory carrying large charges of order of string tension ($\sim \sqrt{\lambda}$) should be dominated at large λ by its semiclassical limit. This observation was used in the past (see, e.g., [1–6]) and was recently applied to computation of 2-point [7–9] and certain 3-point [10–15] correlators of string vertex operators.

The main idea [10, 11] is that a special subset of 3-point correlators containing only two “heavy” operators with large quantum numbers can be computed using the same stationary point trajectory that controls the semiclassical limit of their 2-point function.

In [12] this observation was further generalized and applied to the case when the “light” operator may be representing a string mode (i.e. may not be BPS). It was also suggested [12] that the same approach should apply to higher n -point correlation functions with 2 “heavy” and $n - 2$ “light” operators: the semiclassical expression for n -point correlator should be given by a product of “light” vertex operators computed on the semiclassical world surface determined by the “heavy” operator insertions.

Our aim here will be to study in detail the case of such 4-point functions, providing evidence of consistency of the semiclassical recipe for their computation at strong coupling. Using an independent argument based on differentiating over string tension, we will show, following [12], that the semiclassical expression for the 4-point correlator with 2 integrated dilaton operators can be represented as a product of two 3-point correlation functions, each with one dilaton operator, which matches the semiclassical prescription.

Below we will consider several explicit examples. In particular, we will find an explicit form of the semiclassical 4-point function involving two “heavy” operators corresponding to large-spin folded string in AdS_5 and two “light” chiral primary scalar operators.

We will also consider the case when the 4-point function contains two “heavy” and two “light” chiral primary scalar operators with large charges $\pm J$ and fixed charges $\pm j$ respectively. Since such correlator is close to be extremal [16] for $J \gg j$ one may expect that it may be protected for large J , just like the chiral primary 3-point function is [17]. Indeed, we will find that it exactly matches the $J \gg 1$ limit of the free gauge theory result [18] for the correlator of 4 chiral primary operators with charges $J, -J, 2, -2$.

One may also expect that the semiclassical (large λ , large charge) limit of a correlation function of 4 BPS operators should match the large charge limit of the corresponding supergravity expression computed according to the standard AdS/CFT rules (see, e.g., [20]). Analysing the large J limit of the supergravity expression for the $(J, -J, 2, -2)$ correlator found in [19] we will, however, find a disagreement (the supergravity correlator grows slower with J than its semiclassical or gauge theory counterparts). This issue deserves further investigation.

The rest of the paper is organized as follows. In Section 2 we consider 4-point functions at strong coupling in the case when two operators are “heavy”, i.e. represent semiclassical states, and show that they can be written in a factorized form as a product of two 3-point functions. In Section 3 we give an independent proof of that factorization in the case when the “light” operators are the dilaton operators, thus providing a check of the semiclassical prescription.

In Section 4 we discuss several examples of semiclassical computation of 4-point functions. We start in section 4.1 with reviewing the general form of the chiral primary vertex operator from [10,17,21] and revisit the computation [10,12] of the 3-point correlator of chiral primary scalars in the case when two of them carry large charge. Keeping the AdS_5 boundary positions of the operators arbitrary helps to clarify, in section 4.2, the factorized structure of the semiclassical 4-point function of 2 “heavy” and 2 “light” chiral primary operators. Assuming the charge assignment such that this correlator is an extremal one, we show that this factorized semiclassical structure is in perfect agreement with the non-renormalization conjecture [16] for the extremal correlators. Finally, in section 4.3 we apply the semiclassical method to compute the 4-point function with 2 “heavy” large AdS_5 spin operators and 2 “light” chiral primary scalars.

In section 5 we compare our semiclassical expression for the 4-point function of chiral primary operators with the known free gauge theory [18] and the supergravity [19] expressions for the chiral primary correlator with charges $J, -J, 2, -2$. Taking the large J limit we find perfect agreement with the gauge theory result but an apparent disagreement with the supergravity expression of [19].

Section 6 contains some remarks on consistency between the semiclassical result and general factorization properties of 4-point functions. We also mention some generalizations.

In Appendix A we present the general form of the chiral primary vertex operator of [10,21] and explain when it can be replaced by its simplified form used in [12,14]. In Appendix B we consider the large J limit of an AdS_5 integral entering the supergravity correlation function discussed in section 5.

2 Semiclassical correlation functions in $AdS_5 \times S^5$ with two “heavy” operators

Our object of interest in this paper is 4-point correlation function of string vertex operators dual to gauge invariant local operators with two operators carrying large quantum numbers of order of string tension and the remaining 2 carrying fixed (much smaller) quantum numbers.

We will refer to the former two operators as “heavy” (or “semiclassical”) and to the latter as “light” (or “quantum”). In general, one may consider similar n -point correlators with any number $n - 2 = 0, 1, 2, 3, \dots$ of such “light” operators. We will start with reviewing the case of the two-point [7–9] and three-point correlation functions [10, 12].

The calculation of two-point function in the leading semiclassical approximation was shown in [4, 7, 9] to be intrinsically related to finding an appropriate classical string solution. Let $V_{H1}(\xi_1)$ and $V_{H2}(\xi_2)$ be the two “heavy” vertex operators inserted at points ξ_1 and ξ_2 on the worldsheet (chosen as a plane or a sphere).¹ For large string tension ($\sqrt{\lambda} \gg 1$) two-point function in the semiclassical approximation is dominated by the action evaluated at its stationary point

$$\langle V_{H1}(\xi_1) V_{H2}(\xi_2) \rangle \sim e^{-I}, \quad (2.1)$$

where I is the string action on $AdS_5 \times S^5$ in conformal gauge ($\alpha = 1, 2$)

$$I = \int d^2\xi L, \quad L = \frac{\sqrt{\lambda}}{4\pi} (\partial_\alpha Y_M \partial_\alpha Y^M + \partial_\alpha X_k \partial_\alpha X_k + \text{fermions}), \quad (2.2)$$

$$Y_M Y^M = -Y_0^2 - Y_5^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1, \quad (2.3)$$

$$X_k X_k = X_1^2 + \dots + X_6^2 = 1.$$

The stationary point solution solves the string equation with singular sources, i.e. has singularities prescribed by $V_{H1}(\xi_1)$ and $V_{H2}(\xi_2)$. Using conformal symmetry we can map the ξ -plane to the Euclidean cylinder parametrized by (τ_e, σ)

$$e^{\tau_e + i\sigma} = \frac{\xi - \xi_2}{\xi - \xi_1}. \quad (2.4)$$

It was shown on various examples in flat space and in $AdS_5 \times S^5$ in [4, 7, 9] that under this conformal map the singular solution on ξ -plane transforms into a smooth classical string solution on the cylinder that carries the same quantum numbers (energy, spins, etc.) as the states represented by the vertex operators.

This discussion can be repeated for the physical integrated vertex operators labelled by points \vec{x}_1, \vec{x}_2 on the boundary of the Poincaré patch of AdS_5 [1, 4]

$$V_H(\vec{x}) = \int d^2\xi V_H(\xi; \vec{x}), \quad V_H(\xi; \vec{x}) \equiv V_H(z(\xi), \vec{x}(\xi) - \vec{x}, X_k(\xi)), \quad (2.5)$$

where z and $\vec{x} = (x_0, x_1, x_2, x_3)$ are the Poincaré coordinates of AdS_5 .² The semiclassical two-point function $\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) \rangle$ is again determined by the value of the classical action

¹We consider planar AdS/CFT duality, i.e. only tree-level string theory (world sheets of higher genera correspond to including string or $1/N$ corrections).

²Throughout this paper AdS_5 is always assumed to be Euclidean.

on the stationary point solution. After we perform the conformal map (2.4) we again obtain a smooth solution on the cylinder which is just the corresponding spinning string solution rewritten in Poincaré coordinates which will satisfy the following boundary conditions (see [9] for details)³

$$\tau_e \rightarrow -\infty \implies z \rightarrow 0, \quad \vec{x} \rightarrow \vec{x}_1, \quad \tau_e \rightarrow +\infty \implies z \rightarrow 0, \quad \vec{x} \rightarrow \vec{x}_2. \quad (2.6)$$

Similar method can be applied [10, 12] to the semiclassical computation of three-point functions with two “heavy” and one “light” operators

$$\begin{aligned} G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) &= \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_L(\vec{x}_3) \rangle \\ &= \int \mathcal{D}\mathbb{X}^{\mathbb{M}} e^{-I} \int d^2\xi_1 d^2\xi_2 d^2\xi_3 V_{H1}(\xi_1; \vec{x}_1) V_{H2}(\xi_2; \vec{x}_2) V_L(\xi_3; \vec{x}_3), \end{aligned} \quad (2.7)$$

where $\int \mathcal{D}\mathbb{X}^{\mathbb{M}}$ is the integral over the fields (Y_M, X_k) (as well as fermions which we ignore as we consider only leading-order semiclassical expansion). In the stationary point equations the contribution of the “light” operator can be ignored and then the solution is the same as in the case of two-point function of two “heavy” operators, i.e. it is found by extremising

$$I - \ln V_{H1}(\xi_1; \vec{x}_1) - \ln V_{H2}(\xi_2; \vec{x}_2). \quad (2.8)$$

Here we use that the “heavy” operators carry large charges, so that $\ln V_{H1, H2}$, like I , are proportional to string tension. Then

$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \int d^2\xi V_L(\xi; \vec{x}_3) \int d^2\xi_1 d^2\xi_2 e^{-I} V_{H1}(\xi_1; \vec{x}_1) V_{H2}(\xi_2; \vec{x}_2), \quad (2.9)$$

where I, V_H, V_L are now evaluated on the solution to the equations of motion that follow from (2.8). The second factor in (2.9) is the semiclassical value of the two-point function of the two “heavy” operators. If we divide by it we end up with

$$\frac{G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3)}{G_2(\vec{x}_1, \vec{x}_2)} = \int d^2\xi V_L(z(\xi), \vec{x}(\xi) - \vec{x}_3, X_k(\xi)), \quad (2.10)$$

where $(z(\xi), \vec{x}(\xi), X_k(\xi))$ represents the corresponding string solution carrying the same quantum numbers as the “heavy” vertex operators with the boundary conditions (2.6) transformed to the ξ -plane using (2.4). Using 2d conformal invariance we can also transform (2.10) back to the cylinder to get $(\int d^2\sigma = \int_{-\infty}^{\infty} d\tau_e \int_0^{2\pi} d\sigma)$

$$\frac{G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3)}{G_2(\vec{x}_1, \vec{x}_2)} = \int d^2\sigma V_L(z(\tau_e, \sigma), \vec{x}(\tau_e, \sigma) - \vec{x}_3, X_k(\tau_e, \sigma)). \quad (2.11)$$

³In general, the euclidean solution will not be real and is not required to end on points at the boundary. For example, for an operator representing long folded spinning string the corresponding solution will approach null lines passing through the insertion points [9]. This subtlety will not be important for what follows.

This expression captures the leading dependence on $\sqrt{\lambda} \gg 1$ (the validity of this approximation was discussed in detail in [12]). The global conformal $SO(2, 4)$ symmetry fixes the form of the two-point and three-point functions (we assume that the operators correspond to scalar primaries and $V_{H2} = V_{H1}^*$)

$$G_2(\vec{x}_1, \vec{x}_2) = \frac{C_{12} \delta_{\Delta_1, \Delta_2}}{x_{12}^{\Delta_1 + \Delta_2}}, \quad x_{ij} \equiv |\vec{x}_i - \vec{x}_j|, \quad (2.12)$$

$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{C_{123}}{x_{12}^{\Delta_1 + \Delta_2 - \Delta_3} x_{13}^{\Delta_1 + \Delta_3 - \Delta_2} x_{23}^{\Delta_2 + \Delta_3 - \Delta_1}}, \quad (2.13)$$

where Δ_i are the dimensions of the operators. By choosing the locations x_i appropriately one can remove the dependence on x_{ij} in (2.11) and adapt (2.11) to computing the coefficient C_{123} [10, 12]. Assuming that $\Delta_1 = \Delta_2$ (as their possible difference is subleading in the approximation we consider) we then find (choosing $x_3 = 0$)⁴

$$\frac{G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3 = 0)}{G_2(\vec{x}_1, \vec{x}_2)} = C_{123} \left(\frac{x_{12}}{|x_1| |x_2|} \right)^{\Delta_3}. \quad (2.14)$$

As was suggested in [12], the same logic can be applied to semiclassical computation of any n -point correlation function that contains two “heavy” and n “light” operators. Here we shall focus on the case of the four-point correlator

$$\begin{aligned} G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) &= \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{L1}(\vec{x}_3) V_{L2}(\vec{x}_4) \rangle \\ &= \int \mathcal{D}\mathbb{X}^{\mathbb{M}} e^{-I} \int d^2\xi_1 d^2\xi_2 d^2\xi_3 d^2\xi_4 V_{H1}(\xi_1; \vec{x}_1) V_{H2}(\xi_2; \vec{x}_2) V_{L1}(\xi_3; \vec{x}_3) V_{L2}(\xi_4; \vec{x}_4). \end{aligned} \quad (2.15)$$

The semiclassical trajectory is again the same, i.e. is obtained from (2.8), and to compute the leading semiclassical term in G_4 we need to evaluate the action I and the product of “light” operators on this solution

$$\begin{aligned} &\frac{G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)}{G_2(\vec{x}_1, \vec{x}_2)} \\ &= \int d^2\xi_3 V_{L1}(z(\xi_3), \vec{x}(\xi_3) - \vec{x}_3, X_k(\xi)) \int d^2\xi_4 V_{L2}(z(\xi_4), \vec{x}(\xi_4) - \vec{x}_4, X_k(\xi)) \end{aligned} \quad (2.16)$$

where we divided by the two-point function of the “heavy” operators as in (2.10). Note that the integrals over ξ_3 and ξ_4 decouple from each other, i.e. the four-point function factorizes. Transforming to the (τ_e, σ) -coordinates we get

$$\begin{aligned} \frac{G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)}{G_2(\vec{x}_1, \vec{x}_2)} &= \int d^2\sigma V_{L1}(z(\tau_e, \sigma), \vec{x}(\tau_e, \sigma) - \vec{x}_3, X_k(\tau_e, \sigma)) \\ &\quad \times \int d^2\sigma' V_{L2}(z(\tau'_e, \sigma'), \vec{x}(\tau'_e, \sigma') - \vec{x}_4, X_k(\tau'_e, \sigma')). \end{aligned} \quad (2.17)$$

⁴Here we formally set $C_{12} = 1$ in (2.12), i.e. assumed that the “heavy” operators are normalized. The ratio G_3/G_2 does not, of course, depend on the normalization of the “heavy” operators, i.e. what we will be computing below is, in fact, the invariant ratio C_{123}/C_{12} .

According to (2.11) each integral in (2.17) is the ratio of the three- and two-point functions. Then we obtain the following factorization

$$\begin{aligned} \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{L1}(\vec{x}_3) V_{L2}(\vec{x}_4) \rangle &= \frac{\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{L1}(\vec{x}_3) \rangle \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{L2}(\vec{x}_4) \rangle}{\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) \rangle} \\ G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) &= \frac{G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) G_3(\vec{x}_1, \vec{x}_2, \vec{x}_4)}{G_2(\vec{x}_1, \vec{x}_2)} . \end{aligned} \quad (2.18)$$

This relation has an obvious generalisation to the case of correlators with two “heavy” and many “light” vertex operators.

Let us finish this section with a few comments. The semiclassical expression (2.17), (2.18) for the above 4-point function may be interpreted as describing the process in which a “heavy” classical string emits two “light” quantum strings, each at separate time. The above semiclassical path integral argument is already a sufficient justification that this is a dominant process at large λ . In fact, the process in which the “macroscopic” string first emits one “light” string mode which then decays into two other “light” modes is subleading at large $\sqrt{\lambda}$. As we shall see explicitly in the examples considered below, each of the (normalized) 3-point functions in (2.18) will scale as $\sqrt{\lambda} \gg 1$ while a correlator of 3 “light” states is of order 1.

As is well known, unlike two- and three-point correlators, the x_i dependence of the four-point correlators is not fixed by the conformal invariance: in general, they involve non-trivial functions of the conformal cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} . \quad (2.19)$$

The factorization (2.18) predicted for the leading term in the semiclassical expansion of $\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{L1}(\vec{x}_3) V_{L2}(\vec{x}_4) \rangle$ implies a particular dependence on conformal cross-ratios. Each of the three-point functions in (2.18) is of the form $\langle V_H V_H V_L \rangle$ and its form is fixed by (2.13) but the precise quantum numbers of the “heavy” operators in (2.18), in general, may not coincide with the quantum numbers of the original operators in $G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$ as they may be shifted by the dimensions of the “light” operators. This shift is important because $\langle V_H V_H V_L \rangle$ depends, in particular, on the difference of the dimensions of “heavy” operators (see eq. (2.13)) in which such a shift can produce a leading order contribution. We will see an example of this in Section 4.

3 4-point correlator with two dilaton operators

Let us now follow the discussion in [12] and provide a consistency check of the factorization (2.18) by taking the two “light” operators to be the dilaton operators (with zero S^5 mo-

mentum). In this case it is possible to give an independent derivation of the expression in (2.18).

The vertex operator for the dilaton inserted at the point \vec{x} on the boundary is given by (see, e.g., [12] and refs. there)

$$V_{dil}(\vec{x}) = c_{dil} \int d^2\xi K_4(\xi; \vec{x}) L, \quad K_4(\xi; \vec{x}) = \left(\frac{z}{z^2 + (\vec{x} - \vec{x})^2} \right)^4, \quad (3.1)$$

where c_{dil} is a normalization coefficient and L is the string Lagrangian in (2.2). If we integrate $V_{dil}(\vec{x})$ over the Euclidean 4-space \vec{x} the factor K_4 goes away and we end up with the expression proportional to string action

$$\int d^4x V_{dil}(\vec{x}) = a_{dil} \int d^2\xi L, \quad a_{dil} = \frac{\pi^2}{6} c_{dil}. \quad (3.2)$$

The proportionality coefficient here is independent of \vec{x} and z (this is easy to see by first translating \vec{x} by \vec{x} , then rescaling \vec{x} by z and finally doing the integral).

Let us now consider the general expression for the three-point function involving two “heavy” operators of dimension Δ ($V_{H1} = V_{H2}^*$) and the dilaton,

$$\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) \rangle = \frac{C_{\Delta, dil}}{x_{12}^{2\Delta-4} x_{13}^4 x_{23}^4}, \quad (3.3)$$

and integrate it over \vec{x}_3 . The l.h.s. of (3.3) then gives

$$\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) \int d^4x_3 V_{dil}(\vec{x}_3) \rangle = a_{dil} \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) \int d^2\xi L \rangle. \quad (3.4)$$

Since the averaging is done with the measure e^{-I} where $I = \int d^2\xi L$ is the string action (2.2) containing a factor of $\sqrt{\lambda}$ and thus satisfying $\lambda \frac{\partial}{\partial \lambda} I = \frac{1}{2} I$ the r.h.s. of (3.4) may be written also as

$$\lambda \frac{\partial}{\partial \lambda} \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) \rangle \sim \lambda \frac{\partial \Delta}{\partial \lambda} \frac{\ln(x_{12}^2 \mu^2)}{x_{12}^{2\Delta}}, \quad (3.5)$$

where in the r.h.s. we used (2.13) (μ is a normalization scale or a cutoff). At the same time, integrating the r.h.s. of (3.3) gives

$$\frac{C_{\Delta, dil}}{x_{12}^{2\Delta-4}} \int \frac{d^4x_3}{x_{13}^4 x_{23}^4} \sim C_{\Delta, dil} \frac{\ln(x_{12}^2 \mu^2)}{x_{12}^{2\Delta}}. \quad (3.6)$$

Comparing (3.5) and (3.6) we finish with the following relation (see also [11, 12])⁵

$$C_{\Delta, dil} \sim \lambda \frac{\partial \Delta}{\partial \lambda}. \quad (3.7)$$

⁵Similar argument leading to this relation can be given on the gauge theory side where λ^{-1} appears as a coefficient in from the action and the integrated dilaton operator is proportional to the gauge theory action.

Let us now insert one more dilaton operator and integrate over its position:

$$\begin{aligned} \int d^4x_4 \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) V_{dil}(\vec{x}_4) \rangle &\sim \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) \int d^2\xi L \rangle \\ &\sim \lambda \frac{\partial}{\partial \lambda} \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) \rangle. \end{aligned} \quad (3.8)$$

Using that the three-point function $\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) \rangle$ is determined by (3.3), (3.7) we get

$$\int d^4x_4 \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) V_{dil}(\vec{x}_4) \rangle \sim \lambda \frac{\partial}{\partial \lambda} \left(\frac{\lambda \frac{\partial \Delta}{\partial \lambda}}{x_{12}^{2\Delta-4} x_{13}^4 x_{23}^4} \right). \quad (3.9)$$

Differentiating the bracket we find two terms. The first one comes from differentiating the numerator and is proportional to $\lambda \frac{\partial}{\partial \lambda} (\lambda \frac{\partial \Delta}{\partial \lambda})$. The second term comes from differentiating Δ in the denominator and is of order $(\lambda \frac{\partial \Delta}{\partial \lambda})^2$. In the semiclassical limit of large Δ (scaling as $\lambda^{1/2}$) the first term can be ignored and so we get [12]

$$\int d^4x_4 \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) V_{dil}(\vec{x}_4) \rangle \sim \left(\lambda \frac{\partial \Delta}{\partial \lambda} \right)^2 \frac{\ln(x_{12}^2 \mu^2)}{x_{12}^{2\Delta}} \frac{1}{x_{13}^4 x_{23}^4}. \quad (3.10)$$

From our discussion of the three-point function earlier in this section we already know that this implies that

$$\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) V_{dil}(\vec{x}_4) \rangle \sim \frac{\lambda \frac{\partial \Delta}{\partial \lambda}}{x_{12}^{2\Delta} x_{14}^4 x_{24}^4} \frac{\lambda \frac{\partial \Delta}{\partial \lambda}}{x_{12}^{2\Delta} x_{13}^4 x_{23}^4} x_{12}^{2\Delta}. \quad (3.11)$$

This is precisely the factorized expression in (2.18)

$$\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) V_{dil}(\vec{x}_4) \rangle = \frac{\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_3) \rangle \langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) V_{dil}(\vec{x}_4) \rangle}{\langle V_{H1}(\vec{x}_1) V_{H2}(\vec{x}_2) \rangle}. \quad (3.12)$$

We have thus independently proven (2.18) in the case when the two “light” operators are the dilaton ones. This provides a non-trivial consistency check of the general semiclassical prediction (2.18) as was already mentioned in [12].⁶

The above argument can be easily generalized to the case of more than two dilaton operator insertions. In this case the dominant contribution to the relation like (3.9) will be coming again from the term with maximal power of $\lambda \frac{\partial \Delta}{\partial \lambda}$ which will dominate at strong coupling over terms with multiple derivatives of Δ .

⁶As was discussed at the end of section 2, the contribution to the above 4-point correlator corresponding to the “heavy” state emitting a “light” state (a “graviton”) that then decays into 2 dilatons should be subleading at large λ . Indeed, such contribution may come from the region of the integral over x_4 where x_4 approaches x_3 and would then be proportional to the product of 3-point function with two “heavy” states and one “light” state and three “light” states. As the former will scale as $\sqrt{\lambda}$ while the latter will be of order 1, this contribution will be negligible compared to (3.10) which scales as $(\sqrt{\lambda})^2$ for $\Delta \sim \sqrt{\lambda}$.

4 Explicit form of semiclassical correlators involving chiral primary and twist-two operators

In this section we will find explicit form of some four-point correlation functions in the leading semiclassical limit. We will start with the case of four chiral primary operators (CPO's). To understand the factorization (2.18) in detail we need to revisit their three-point function [10, 12].

4.1 3-point function of chiral primary operators revisited

Let us consider the three-point function

$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \langle V_{-J}(\vec{x}_1) V_{J-j}(\vec{x}_2) V_j(\vec{x}_3) \rangle, \quad J \gg j. \quad (4.1)$$

Below we shall label vertex operators by their charges (or spins) rather than dimensions. Here V_j stands for a “light” chiral primary operator with fixed charge (S^5 angular momentum) j while V_J is its “heavy” counterpart with large charge $J \sim \sqrt{\lambda}$. In the leading semiclassical approximation we may assume that $V_{J-j} \approx V_J$ but it will be useful to keep this distinction (and thus have manifest charge conservation) in a part of the discussion that follows.

According to (2.11) in the limit of large J this three-point function is given by

$$\frac{G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3)}{G_2(\vec{x}_1, \vec{x}_2)} = \int d^2\sigma V_j(z(\tau_e, \sigma), \vec{x}(\tau_e, \sigma) - \vec{x}_3, X_k(\tau_e, \sigma)). \quad (4.2)$$

Here $z(\tau_e, \sigma)$, $\vec{x}(\tau_e, \sigma)$ and $X_k(\tau_e, \sigma)$ correspond to a point-like string orbiting big circle of S^5 ; the corresponding Euclidean trajectory in the Poincaré patch of AdS_5 satisfies the boundary conditions (2.6). Since the three-point function should depend only on the absolute values of the coordinate differences, without loss of generality we can choose all the points to lie along the x_{0e} -axis. Let us denote the zeroth components of \vec{x}_1 and \vec{x}_2 as a_1 and a_2 respectively. For concreteness we will assume that $a_1 > a_2$. Then the corresponding stationary-point solution is given by [9] ($a_{12} \equiv a_1 - a_2$)

$$\begin{aligned} z &= \frac{a_{12}}{2 \cosh(\kappa \tau_e)}, & x_{0e} &= \frac{a_{12}}{2} \tanh(\kappa \tau_e) + \frac{1}{2}(a_1 + a_2), \\ x_1 &= x_2 = x_3 = 0, \\ \phi &= -i\nu\tau_e, & J &= \sqrt{\lambda} \nu, & \kappa &= \nu, \end{aligned} \quad (4.3)$$

where ϕ is the angle of S^5 .

The expression for the “light” chiral primary vertex operator can be obtained starting from the general expression for the $SO(2, 4)$ covariant 10-d graviton “wave function” [10,

17, 21]. As we show in Appendix A, it can be put in the following form (for simplicity, we choose the location of the operator to be at $\vec{x}_3 = 0$, but the dependence on \vec{x}_3 can be easy to restore at the end)⁷

$$V_j(0) = \hat{c}_j \int d^2\sigma K_j X^j U , \quad (4.4)$$

$$K_j = \left(\frac{z}{z^2 + \vec{x}^2} \right)^j , \quad X \equiv X_1 + iX_2 , \quad \hat{c}_j = \frac{1}{N} \frac{\sqrt{\lambda}}{8\pi} (j+1) \sqrt{j} . \quad (4.5)$$

The operator U here has the following structure

$$U = U_1 + U_2 + U_3 , \quad (4.6)$$

$$U_1 = \frac{1}{z^2} \left[(\partial_\alpha \vec{x})^2 - (\partial_\alpha z)^2 \right] - (\partial_\alpha X_k)^2 , \quad (4.7)$$

$$U_2 = \frac{8}{(z^2 + \vec{x}^2)^2} \left[\vec{x}^2 (\partial_\alpha z)^2 - (\vec{x} \cdot \partial_\alpha \vec{x})^2 \right] , \quad U_3 = \frac{8(\vec{x}^2 - z^2)}{z(z^2 + \vec{x}^2)^2} (\vec{x} \cdot \partial_\alpha \vec{x}) \partial_\alpha z . \quad (4.8)$$

Let us now evaluate (4.4) on the solution (4.3). We get $X^j = e^{ij\phi} = e^{j\kappa\tau_e}$ and (see Appendix A for details)

$$U_1 = \frac{2\kappa^2}{\cosh^2 \kappa\tau_e} , \quad U_2 + U_3 = -\frac{2\kappa^2}{\cosh^2 \kappa\tau_e} \frac{(a_1^2 - a_2^2)(a_1^2 e^{2\kappa\tau_e} - a_2^2 e^{-2\kappa\tau_e})}{(a_1^2 e^{\kappa\tau_e} + a_2^2 e^{-\kappa\tau_e})^2} . \quad (4.9)$$

Substituting this into (4.4) we obtain

$$\frac{G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3 = 0)}{G_2(\vec{x}_1, \vec{x}_2)} = 16\pi\kappa^2 \hat{c}_j \int_{-\infty}^{\infty} d\tau_e \frac{a_1^2 a_2^2 a_{12}^j e^{j\kappa\tau_e}}{(a_1^2 e^{\kappa\tau_e} + a_2^2 e^{-\kappa\tau_e})^{j+2}} . \quad (4.10)$$

After performing the integral (first rescaling $\kappa\tau_e \rightarrow \tau_e$ and then shifting $\tau_e \rightarrow \tau_e + \ln \frac{a_2}{a_1}$) we find

$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3 = 0) = \frac{C_{J,j}}{x_{12}^{2J-j} |\vec{x}_1|^{2j}} \approx \frac{C_{J,j}}{x_{12}^{2J} |\vec{x}_1|^{2j}} , \quad (4.11)$$

where

$$C_{J,j} = \hat{c}_j \frac{8\pi\kappa}{j+1} = \frac{1}{N} J \sqrt{j} . \quad (4.12)$$

We have restored the Lorentz invariance by replacing a_1 and a_{12} with $|\vec{x}_1|$ and x_{12} (recall that in (2.13) we defined $x_{ij} \equiv |\vec{x}_i - \vec{x}_j|$). To restore the dependence on non-zero \vec{x}_3 in the right hand side of (4.11) we may simply replace $|\vec{x}_1|$ with x_{13} .

Note that the dependence on x_{ij} in (4.11) came out to be consistent with (2.13) with

$$\Delta_1 + \Delta_2 \simeq 2J , \quad \Delta_1 - \Delta_2 = \Delta_3 = j \quad (4.13)$$

⁷Here we normalize the chiral primary scalar operator as in [10, 21]. The $\frac{\sqrt{\lambda}}{2\pi}$ factor in \hat{c}_j is the string tension (the graviton operator is a perturbation of the graviton coupling term in the string action) while $1/N$ stands for a formal factor of string coupling ($N \gg 1$ is the rank of the gauge group of the dual gauge theory with $1/N$ being the standard normalization of planar 3-point functions).

which implies, for $J \gg j$, that

$$\Delta_1 = J, \quad \Delta_2 = J - j, \quad \Delta_3 = j \quad (4.14)$$

and is thus consistent with the charge conservation.

Note that if we took the charge of the “light” operator to be $-j$ instead of j we would get a similar expression with the same $C_{J,j}$ and x_1 and x_2 interchanged

$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3 = 0) = \frac{C_{J,j}}{x_{12}^{2J} |\vec{x}_2|^{2j}}, \quad (4.15)$$

with the x_3 -dependence restored again by $|\vec{x}_2| \rightarrow x_{23}$. The latter expression corresponds to the three-point function of operators of dimensions $\Delta_1 = J$, $\Delta_2 = J + j$, $\Delta_3 = j$, i.e. with the charges $-J, J + j, -j$. These observations about the precise charges of the participating chiral primary operators will be important in the next subsection where we will discuss their four-point function.

It is useful to note that $U_2 + U_3$ in (4.9) vanishes if $a_1^2 - a_2^2 = 0$, i.e. for the choice of $a_1 = -a_2$. Computing the 3-point correlator (4.2) in the particular case of $x_1 = -x_2$ one can then use the “simplified” version of the vertex operator (4.4) with U given just by U_1 in (4.7). This operator was used in [10, 12].⁸

4.2 4-point function of two “heavy” and two “light” chiral primary operators

Let us now apply the results of the previous section to compute the four-point function of the chiral primary operators carrying charges $(-J, J, -j, j)$ in the limit of large $J \gg j$. According to the semiclassical prescription (2.16) it is given by

$$\frac{G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)}{G_2(\vec{x}_1, \vec{x}_2)} = V_{-j}(\vec{x}_3) \cdot V_j(\vec{x}_4), \quad (4.16)$$

where $V_{\mp j}(\vec{x}_{3,4})$ are given by (4.4) with \vec{x} shifted by $\vec{x}_{3,4}$ and understood to be evaluated on the solution (4.3). All the necessary ingredients were found in the previous subsection and now we only need to collect them to arrive at the result

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \frac{\langle V_{-J}(\vec{x}_1) V_{J+j}(\vec{x}_2) V_{-j}(\vec{x}_3) \rangle \langle V_{-J}(\vec{x}_1) V_{J-j}(\vec{x}_2) V_j(\vec{x}_4) \rangle}{\langle V_{-J}(\vec{x}_1) V_J(\vec{x}_2) \rangle}. \quad (4.17)$$

⁸In [10] the justification for its use instead of the full vertex of [17, 21] was based on considering the special case when the insertion point \vec{x}_3 is sent to ∞ (which is, in general, enough due to conformal invariance). In this case the addition terms U_2, U_3 in (4.8) are suppressed. In [12] the three-point correlator was chosen to have $a_1 = -a_2$ and in this case again the additional contributions vanish as was explained above.

Note that the charges of the “heavy” operators appearing in the three-point functions in (4.17) are formally different from their original charges $(-J, J)$, in the four-point function. As was already mentioned at the end of Section 2, it is important to keep track of the precise charges when writing factorised expressions in terms of the three-point functions. The right hand side of eq. (4.2) computes the leading contribution to the ratio of the three- and two-point functions in the limit of large λ and J . This leading contribution depends not only on the dimensions of the “heavy” operators but also on the difference of their dimensions which is sensitive to the shift of J by $\pm j$. On the other hand, the structure constant $C_{J,j}$ depends only on the dimensions of the operators and not on their difference. Hence, in computing $C_{J,j}$ such a shift is not important. Note also that both three-point functions in (4.17) are consistent with charge conservation.

Using (4.11)–(4.15) we may write (4.17) in a more explicit form as

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \frac{jJ^2}{N^2} \frac{1}{x_{12}^{2J} x_{23}^{2j} x_{14}^{2j}}. \quad (4.18)$$

It is convenient also to present $G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$ in a form involving cross-ratios defined in (2.19)

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \frac{1}{x_{12}^{2J} x_{34}^{2j}} \mathcal{F}(u, v), \quad \mathcal{F}(u, v) = \frac{jJ^2}{N^2} \frac{u^j}{v^j}, \quad (4.19)$$

where we used that $J \gg j$.

Let us now consider a slightly different 4-point correlator of chiral primary operators – with charges $(-J - j_1 - j_2, J, j_1, j_2)$, where $j_1, j_2 \geq 0$. Like similar 3-point correlators considered above, this is an extremal correlator as $\Delta_{-J-j_1-j_2} = J + j_1 + j_2 = \Delta_J + \Delta_{j_1} + \Delta_{j_2}$. For such extremal correlators of BPS operators with $\Delta_1 = \Delta_2 + \dots + \Delta_n$ there exists a non-renormalization conjecture [16, 20] (here for generality we label operators by their dimensions and consider only planar approximation)

$$\langle V_{\Delta_1}(\vec{x}_1) V_{\Delta_2}(\vec{x}_2) \dots V_{\Delta_n}(\vec{x}_n) \rangle = \frac{A(\{\Delta_i\})}{N^{n-2}} \prod_{k=2}^n \frac{1}{x_{1k}^{2\Delta_k}}, \quad (4.20)$$

where the coefficient $A(\{\Delta_i\})$ should not depend on the 't Hooft coupling λ , i.e. should be the same at weak and strong coupling. According to the semiclassical prescription (2.16) for the above choice of charges of chiral primary operators we again obtain the expression in eq. (4.16). Since both j_1 and j_2 are assumed to be positive we then find using (4.11) and (4.12)

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \frac{J^2 \sqrt{j_1 j_2}}{N^2} \frac{1}{x_{12}^{2J} x_{13}^{2j_1} x_{14}^{2j_2}}. \quad (4.21)$$

This expression is in perfect agreement with (4.20) which is not too surprising as this 4-point correlator is expressed in terms of extremal 3-point correlators. This observation

provides another consistency check of the semiclassical method of computing such higher-point correlation functions.

4.3 4-point function of two large spin S operators and two “light” chiral primary operators

Let us now consider an example of 4-point correlator involving two non-BPS “heavy” operators dual to large spin gauge theory operators and two “light” chiral primary operators. The minimal twist large spin gauge theory operators are dual to a folded string with spin S in AdS_3 [2]. We will denote the corresponding vertex operator as V_S , with $V_{-S} \equiv V_S^*$. The two-point function of such operators can be computed semiclassically in the limit of large $S \gg \sqrt{\lambda}$ spin [7, 9] giving (we assume that V_S is normalized appropriately)

$$\langle V_S(\vec{x}_1) V_{-S}(\vec{x}_2) \rangle = \frac{1}{x_{12}^{2\Delta(S)}}, \quad (4.22)$$

$$\Delta(S) = S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \dots \quad (4.23)$$

The corresponding 3-point functions with two large spin S and one BPS operator were already considered in [12]. Let us consider the 3-point function with one “light” chiral primary operator

$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \langle V_S(\vec{x}_1) V_{-S}(\vec{x}_2) V_j(\vec{x}_3) \rangle. \quad (4.24)$$

For simplicity, we will set $\vec{x}_3 = 0$. The 3-point function coefficients are then determined by the ratio $G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3)/G_2(\vec{x}_1, \vec{x}_2)$ in eq. (4.4). Compared to [12] here we start with generic positions \vec{x}_1 , \vec{x}_2 and thus need to use the general form of the chiral primary vertex in (4.6), (4.8). The integral in (4.4) is to be evaluated on the euclidean stationary point solution corresponding to the long folded spinning string in AdS_3 [9] (cf. (4.3))⁹

$$z = \frac{a_{12}}{2 \cosh(\kappa\tau_e) \cosh(\mu\sigma)}, \quad x_{0e} = \frac{a_{12}}{2} \tanh(\kappa\tau_e) + \frac{1}{2}(a_1 + a_2), \quad (4.25)$$

$$x_1 + ix_2 = \frac{a_{12} \tanh(\mu\sigma)}{2 \cosh(\kappa\tau_e)} e^{i\varphi}, \quad \varphi = -i\kappa\tau_e, \quad x_3 = 0,$$

$$\kappa = \mu = \frac{1}{\pi} \ln \mathcal{S}, \quad \mathcal{S} = \frac{S}{\sqrt{\lambda}} \gg 1. \quad (4.26)$$

This solution (4.25) is valid in the limit of large \mathcal{S} on the interval $\sigma \in [0, \frac{\pi}{2}]$, describing one quarter of the folded string. If we choose $a_1 = -a_2$ then on this solution $U_2 + U_3 = 0$, i.e. the

⁹As in the subsection 4.1, we can choose without loss of generality, the points \vec{x}_1 and \vec{x}_2 to lie on the x_{0e} -axis as the three-point function should depend only on the absolute values of the coordinate differences. As above, we again denote the zeroth components of \vec{x}_1 and \vec{x}_2 by a_1 and a_2 .

chiral primary vertex operator takes its simplified form with $U = U_1$ used in [12]. Keeping a_1 and a_2 arbitrary we then get from (4.4), (4.6), (4.8) (see Appendix A for details)

$$\frac{G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3 = 0)}{G_2(\vec{x}_1, \vec{x}_2)} = 4\hat{c}_j \int_0^{\pi/2} d\sigma \int_{-\infty}^{\infty} d\tau_e \frac{2a_{12}^j}{\cosh^j(\mu\sigma) (a_1^2 e^{\kappa\tau_e} + a_2^2 e^{-\kappa\tau_e})^j} \times \left[\frac{4\kappa^2 a_1^2 a_2^2}{(a_1^2 e^{\kappa\tau_e} + a_2^2 e^{-\kappa\tau_e})^2} - \mu^2 \tanh^2(\mu\sigma) \right]. \quad (4.27)$$

Changing the variables $\kappa\tau_e \rightarrow \tau_e$, $\mu\sigma = \kappa\sigma \rightarrow \sigma$ and $\tau_e \rightarrow \tau_e + \ln \frac{a_2}{a_1}$ we can pull all the dependence on x_i out of the integral to get

$$G_3(\vec{x}_1, \vec{x}_2, \vec{x}_3 = 0) = \frac{C_{S,j}}{x_{12}^{2\Delta(S)} |\vec{x}_1|^j |\vec{x}_2|^j}, \quad (4.28)$$

where

$$C_{S,j} = 8\hat{c}_j \int_0^{\frac{1}{2}\pi\kappa} d\sigma \int_{-\infty}^{\infty} d\tau_e \frac{1}{\cosh^j \sigma (e^{\tau_e} + e^{-\tau_e})^j} \left(\frac{1}{\cosh^2 \tau_e} - \tanh^2 \sigma \right) \quad (4.29)$$

and we have replaced a_1 , a_2 and a_{12} with the Lorentz invariant objects $|\vec{x}_1|$, $|\vec{x}_2|$ and x_{12} . Evaluating the integrals in the limit of large κ we find that the leading term in $C_{S,j}$ is constant (i.e. does not depend on $\kappa = \frac{1}{\pi} \ln \mathcal{S}$)

$$C_{S,j} \approx \hat{c}_j \frac{4\pi\Gamma[\frac{j}{2}]^2}{2^j\Gamma[\frac{j-1}{2}]\Gamma[\frac{j+3}{2}]} = \frac{1}{N} \sqrt{\lambda} \frac{\sqrt{j}}{2^j} \frac{\Gamma[\frac{j}{2}]^2}{\Gamma[\frac{j-1}{2}]\Gamma[\frac{j+1}{2}]}, \quad (4.30)$$

which is the same as the expression found in [12] (the leading term in eq.(4.28) there).

To restore the \vec{x}_3 dependence in (4.28) we should again replace x_1 with x_{13} and x_2 with x_{23} . The structure of (4.28) is then the expected one for the three conformal operators with dimensions

$$\Delta_1 = \Delta_2 = \Delta(S), \quad \Delta_3 = j. \quad (4.31)$$

Let us note that so far we ignored the issue of S^5 angular momentum conservation as its effect is subleading at large S . We may explicitly maintain the S^5 momentum conservation by considering the operator with the charges (S, j_1) , $(-S, -j-j_1)$, $(0, j)$. The additional j, j_1 dependent terms correcting (4.28),(4.30) will be suppressed by a factor of $\mu^{-1} \sim (\ln \mathcal{S})^{-1} \ll 1$, see [12, 14], so that the leading term will not depend on them.

Consider now the four-point function¹⁰

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \langle V_S(\vec{x}_1) V_{-S}(\vec{x}_2) V_{j_1}(\vec{x}_3) V_{j_2}(\vec{x}_4) \rangle. \quad (4.32)$$

¹⁰We may assume $j_2 = -j_1$ to satisfy explicitly the charge conservation. Then $x_{ij}^{j_2}$ below should be replaced by $x_{ij}^{|j_2|}$.

According to (2.17) for $S \gg j_1, j_2$ it is then given by the product of two chiral primary vertex operators evaluated on the solution (4.25). Using (4.22) we then obtain

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \frac{C_{S,j_1} C_{S,j_2}}{x_{12}^{2\Delta(S)} x_{13}^{j_1} x_{23}^{j_1} x_{14}^{j_2} x_{24}^{j_2}} . \quad (4.33)$$

We can also present this in the form of (2.18)

$$G_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \frac{\langle V_S(\vec{x}_1) V_{-S}(\vec{x}_2) V_{j_1}(\vec{x}_3) \rangle \langle V_S(\vec{x}_1) V_{-S}(\vec{x}_2) V_{j_2}(\vec{x}_4) \rangle}{\langle V_S(\vec{x}_1) V_{-S}(\vec{x}_2) \rangle} \quad (4.34)$$

Let us finish with two comments.

The 3-point coefficient in (4.30) is proportional to the string tension, i.e. $\sim \sqrt{\lambda} \gg 1$ (this factor originates from the normalization constant in the “light” vertex operator in (4.5) since it is determined by a “graviton” perturbation in the string action). The same is also true for the BPS 3-point function in (4.12) where $J = \sqrt{\lambda} \kappa \sim \sqrt{\lambda}$. As a result, the semiclassical 4-point functions in (4.21) or in (4.33) scale as $(\sqrt{\lambda})^2$.

The expression (4.30) implies also that for large $j \gg 1$ (but with $\frac{j}{\sqrt{\lambda}} \ll 1$ for the validity of our approximation) one finds $C_{S,j} \rightarrow \frac{\sqrt{\lambda}}{N} \sqrt{j} \exp(-j \ln 2)$. Such exponential suppression is expected for correlators in which all charges of the vertex operators are large so that such correlators should be dominated by a semiclassical action factor $\sim e^{-a\sqrt{\lambda}}$ (cf. [8, 15]). The same applies then also to the 4-point function (4.33).

5 Comparison of semiclassical correlator of four chiral primary operators with the free gauge theory and supergravity results

Let us now compare our semiclassical result (4.18),(4.19) for the correlator of two “heavy” and two “light” chiral primary operators with results available in the literature. To be able to take the semiclassical limit we need a correlator of BPS operators with two arbitrary charges that can be taken to be large. As far as we know, the only such correlator that was considered in the literature is that with two chiral primary operators of arbitrary dimension J and two of dimension 2 [18, 19].

To facilitate comparison with the expressions in [18, 19] we find it convenient to relabel the coordinates $\vec{x}_1 \leftrightarrow \vec{x}_4$, $\vec{x}_2 \leftrightarrow \vec{x}_3$ in (4.19). Note that the definitions of u, v are kept the same as in (2.19). Then for $j = 2$ the expression in (4.19) becomes

$$\langle V_2(\vec{x}_1) V_{-2}(\vec{x}_2) V_J(\vec{x}_3) V_{-J}(\vec{x}_4) \rangle_{semicl} = \frac{1}{x_{12}^4 x_{34}^{2J}} \frac{2J^2 u^2}{N^2 v^2} . \quad (5.1)$$

This semiclassical result is found in the limit $\sqrt{\lambda} \gg 1$ and $J \sim \sqrt{\lambda} \gg 1$. Since it is given by a product of the three-point chiral primary correlators divided by their two-point function each of which is not [17, 22] renormalized by λ we may conjecture that this expression should match the large J limit of the corresponding gauge theory correlator in free $\mathcal{N}=4$ SYM theory. Also, for large charges such 4-point correlator approaches an extremal one ($J \approx J + 2 + 2$) and thus, as was mentioned at the end of section 4.2, should not be renormalized.

In free $SU(N)$ SYM theory the four-point function of chiral primary gauge theory operators $\mathcal{O}_j(\vec{x}, n)$ defined as (Φ^k are 6 real scalars and $n = (n_1, \dots, n_6)$ is complex constant vector)

$$\mathcal{O}_j(\vec{x}, n) = \frac{1}{\sqrt{j}} \left(\frac{8\pi^2}{\lambda} \right)^{j/2} n_{k_1} \dots n_{k_j} \text{tr}(\Phi^{k_1} \dots \Phi^{k_j}), \quad \mathcal{O}_{-j}(\vec{x}, n) = \mathcal{O}_j(\vec{x}, \bar{n}), \quad (5.2)$$

$$\sum_{k=1}^6 n_k n_k = 0, \quad \sum_{k=1}^6 n_k \bar{n}_k = 1, \quad (5.3)$$

was computed in [18] for the choice of operators with charges $2, -2, J, -J$ (see also [19]). For $J \geq 4$ it is given by (we consider the planar limit $N \gg 1$)

$$\begin{aligned} & \langle \mathcal{O}_2(\vec{x}_1, n_1) \mathcal{O}_{-2}(\vec{x}_2, n_2) \mathcal{O}_J(\vec{x}_3, n_3) \mathcal{O}_{-J}(\vec{x}_4, n_4) \rangle_{\text{free gauge th.}} \\ &= \left(\frac{n_1 \cdot n_2}{x_{12}^2} \right)^2 \left(\frac{n_3 \cdot n_4}{x_{34}^2} \right)^J \mathcal{G}^{2,2,J,J}(u, v; p, q) \end{aligned} \quad (5.4)$$

Here u and v are again the conformal cross-ratios (2.19) and the function $\mathcal{G}^{2,2,J,J}(u, v; p, q)$ is given by

$$\mathcal{G}^{2,2,J,J}(u, v; p, q) = 1 + \frac{2J}{N^2} \left[pu + q \frac{u}{v} + (J-1) \left(pq \frac{u^2}{v} + p^2 u^2 + q^2 \frac{u^2}{v^2} \right) \right], \quad (5.5)$$

where p and q are defined as

$$p = \frac{(n_1 \cdot n_3)(n_2 \cdot n_4)}{(n_1 \cdot n_2)(n_3 \cdot n_4)}, \quad q = \frac{(n_1 \cdot n_4)(n_2 \cdot n_3)}{(n_1 \cdot n_2)(n_3 \cdot n_4)}. \quad (5.6)$$

The term 1 in (5.5) is the disconnected contribution that we ignored in the previous discussion, i.e. $\langle \mathcal{O}_2(\vec{x}_1, n_1) \mathcal{O}_{-2}(\vec{x}_2, n_2) \rangle \langle \mathcal{O}_J(\vec{x}_3, n_3) \mathcal{O}_{-J}(\vec{x}_4, n_4) \rangle$; it is useful to keep it here to indicate that the two-point function is assumed to be unit-normalized.

Let us specify these expressions to the operators of interest

$$\mathcal{O}_2(\vec{x}_1) = \frac{1}{\sqrt{2}} \frac{4\pi^2}{\lambda} \text{tr} Z^2, \quad \mathcal{O}_{-2}(\vec{x}_2) = \frac{1}{\sqrt{2}} \frac{4\pi^2}{\lambda} \text{tr} \bar{Z}^2, \quad Z \equiv \Phi^1 + i\Phi^2, \quad (5.7)$$

$$\mathcal{O}_J(\vec{x}_3) = \frac{1}{\sqrt{J}} \left(\frac{4\pi^2}{\lambda} \right)^{J/2} \text{tr} Z^J, \quad \mathcal{O}_{-J}(\vec{x}_4) = \frac{1}{\sqrt{J}} \left(\frac{4\pi^2}{\lambda} \right)^{J/2} \text{tr} \bar{Z}^J, \quad (5.8)$$

corresponding to the following choice of \mathbf{n}_r

$$\mathbf{n}_1 = \mathbf{n}_3 = \frac{1}{\sqrt{2}}(1, i, 0, 0, 0, 0), \quad \mathbf{n}_2 = \mathbf{n}_4 = \frac{1}{\sqrt{2}}(1, -i, 0, 0, 0, 0). \quad (5.9)$$

Then (5.4) simplifies to

$$\langle \mathcal{O}_2(\vec{x}_1) \mathcal{O}_{-2}(\vec{x}_2) \mathcal{O}_J(\vec{x}_3) \mathcal{O}_{-J}(\vec{x}_4) \rangle_{free\ gauge\ th.} = \frac{1}{x_{12}^4 x_{34}^{2J}} \left[1 + \frac{2J}{N^2} \frac{u}{v} + \frac{2J(J-1)}{N^2} \frac{u^2}{v^2} \right]. \quad (5.10)$$

If we now take J to be large the dominant contribution to connected $O(\frac{1}{N^2})$ part of the correlator comes from the last term in (5.10), i.e. we match precisely our semiclassical prediction at strong coupling (5.1).

Next, let us attempt to compare (5.1) with the supergravity result which should also correspond to strong coupling limit of planar gauge theory correlator with chiral primary operators represented by the appropriate [17, 23] supergravity scalar modes. Here one finds [19]

$$\langle \mathcal{O}_2(\vec{x}_1) \mathcal{O}_{-2}(\vec{x}_2) \mathcal{O}_J(\vec{x}_3) \mathcal{O}_{-J}(\vec{x}_4) \rangle_{supergr} = \frac{1}{x_{12}^4 x_{34}^{2J}} \left[1 + \frac{2J}{N^2} \frac{u}{v} - \frac{1}{N^2} \hat{D}(u, v; J) \right] \quad (5.11)$$

$$\hat{D}(u, v; J) = \frac{2Ju^J}{(J-2)!} \bar{D}_{J, J+2, 2, 2}(u, v). \quad (5.12)$$

Here the function \bar{D} is defined as follows in terms of the standard four scalar bulk-to-boundary propagator integral in AdS_5 (with dimensions of operators being $\Delta_1, \dots, \Delta_4$)¹¹

$$\bar{D}_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(u, v) = c_\Gamma \frac{x_{13}^{2\Sigma-2\Delta_4} x_{24}^{2\Delta_2}}{x_{14}^{2\Sigma-2\Delta_1-2\Delta_4} x_{34}^{2\Sigma-2\Delta_3-2\Delta_4}} D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}, \quad (5.13)$$

$$D_{\Delta_1, \Delta_2, \Delta_3, \Delta_4}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \int \frac{dz d^4x}{z^5} \prod_{i=1}^4 \left[\frac{z}{z^2 + (\vec{x} - \vec{x}_i)^2} \right]^{\Delta_i}, \quad (5.14)$$

$$\Sigma \equiv \frac{1}{2} \sum_{i=1}^4 \Delta_i, \quad c_\Gamma = \frac{2}{\pi^2} \frac{\Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3) \Gamma(\Delta_4)}{\Gamma(\Sigma - 2)}. \quad (5.15)$$

The first two terms in (5.11) are the same as in the free gauge theory expression (5.10) but instead of the third term in (5.10) we have a complicated \hat{D} -term. As we will show in Appendix B, the \hat{D} -term is *subleading* for large J , i.e. for large J the supergravity expression is dominated by the second ($\frac{2J}{N^2} \frac{u}{v}$) term. This is in an apparent contradiction with our semiclassical result (5.1).

¹¹The expression in (5.12) is a result of a non-trivial summation of many contributions which is the reason why \bar{D} has a somewhat “counter-intuitive” assignment of its labels compared to order of the operators in the l.h.s. in (5.11).

At the same time, the correspondence between the supergravity computation and the string semiclassical computation for “massless” string modes (e.g., the supergravity chiral primary scalars) should be expected on general grounds. Indeed, the supergravity computation in terms of scalar particle propagators should admit a reformulation in terms of “first-quantized” superparticle path integral and 1-d vertex operators, as it should be an appropriate $\alpha' \sim \frac{1}{\sqrt{\lambda}} \rightarrow 0$ limit of the full string computation.¹² This agreement would be restored if the supergravity expression in (5.11) found in [19] is, in fact, missing a term equivalent to the last term in the free gauge theory result (5.10), i.e.

$$\left(\langle \mathcal{O}_2(\vec{x}_1) \mathcal{O}_{-2}(\vec{x}_2) \mathcal{O}_J(\vec{x}_3) \mathcal{O}_{-J}(\vec{x}_4) \rangle \right)_{\text{supergr. extra}} = \frac{1}{x_{12}^4 x_{34}^{2J}} \frac{2J(J-1)}{N^2} \frac{u^2}{v^2}. \quad (5.16)$$

This term will then dominate at large J and match the semiclassical expression in (5.1).¹³

6 Discussion

As was discussed above, the semiclassical expression (2.17),(2.18) for the 4-point functions may be interpreted as describing the process in which a “heavy” classical closed string emits two “light” quantum string modes. This can be pictured as a cylindrical world surface with two “light” world lines attached at different points. The process where the “macroscopic” string first emits one “light” mode which then decays into two other “light” modes. The reason is that the latter process. Indeed, each 3-point function entering (2.18) contains two “heavy” operators and thus scales as $\sqrt{\lambda}$ while a correlator of 3 “light” states will be of order 1.¹⁴

In general, either on the string side or on the dual CFT side, higher-point correlators of primary conformal operators are, in principle, determined (via the associative OPE or factorization) by their 2-point and 3-point correlation functions. Considering the correlator

¹²An apparent difference is that in the semiclassical string theory discussion we took $\sqrt{\lambda} \gg 1$ with $\frac{J}{\sqrt{\lambda}}$ fixed while in the supergravity analysis J can take any integer values. However, here we are dealing only with massless string modes (or BPS states) and in both cases $\sqrt{\lambda}$ is taken to infinity, and also we are after the leading (positive-power) $J^2 \gg 1$ contribution, there is no reason to doubt that the supergravity expression should be valid for all values of J , including $J \sim \sqrt{\lambda} \rightarrow \infty$. The above arguments suggesting non-renormalization of this particular correlator support this expectation.

¹³It would be important to recheck the computation of [19] since comparison of our result with the supergravity one is a priori non-trivial. In the computation based directly on the supergravity action, one has to sum, even in the limit of large J , over many Witten’s diagrams corresponding to exchanges of BPS states with large charges. The string theory computation based on vertex operators non-trivially “repackage” the field theory result, even when applied to “protected” correlators.

¹⁴For example, for 3 chiral primary scalars with charges j_i one gets $G_3 = \frac{1}{N} \sqrt{j_1 j_2 j_3}$ [17].

in (2.15) we may factorize it, say, in $13 \rightarrow 24$ channel, getting, symbolically

$$\langle V_H V_H^* V_{L1} V_{L2} \rangle = \sum_A c_A \langle V_H V_{L1} V_A^* \rangle \langle V_A V_H^* V_{L2} \rangle . \quad (6.1)$$

In the semiclassical (large charge, large $\sqrt{\lambda}$) approximation the sum over the intermediate states will be dominated by the contribution with $V_A = V_H$, i.e. will be given by the expression in (2.18). The same result should be found also if we factorise in the $12 \rightarrow 34$ channel, i.e.

$$\langle V_H V_H^* V_{L1} V_{L2} \rangle = \sum_A c_A \langle V_H V_H^* V_A^* \rangle \langle V_A V_{L1} V_{L2} \rangle . \quad (6.2)$$

To reproduce the semiclassical result (2.18) from (6.2) one would need to sum over an infinite number of contributions of intermediate states. One particular type of contributions in (6.2) will be the one with V_A as a “light” state, which, as was mentioned above, is subleading in the semiclassical limit.

The terms in (6.2) with V_A as a “heavy” state, i.e. $V_A = V_{H'}$, should also be subleading for large $\sqrt{\lambda}$. Indeed, $\langle V_H V_H^* V_{H'} \rangle$ and $\langle V_{H'} V_{L1} V_{L2} \rangle$ are expected to scale exponentially in $\sqrt{\lambda}$ at large $\sqrt{\lambda}$.¹⁵ Hopefully, their explicit semiclassical expressions will be possible to find using integrability-based method recently developed in [15].¹⁶

One may also consider 4-point functions involving more than 2 “heavy” operators. One might expect that $\langle V_{H1} V_{H2} V_{H3} V_L \rangle$ will again scale exponentially with large $\sqrt{\lambda}$. At the same time, the factorization relations similar to (6.1), (6.2) seem to suggest that at least for $x_1 \rightarrow x_2$, $x_3 \rightarrow x_4$ (and particular choice of large charges) the 4-point correlator

¹⁵The leading semiclassical contribution is given by e^{-I} where I is the string action evaluated on the classical solution. Strictly speaking, it is not clear a priori if the exponential factor will always decay with large $\sqrt{\lambda}$ as the value of the string action on the stationary-point (possibly complex) Euclidean solution may not be positive. Still, the expectation of decay is supported by the explicit example in [15] of 3 “heavy” operators (one non-BPS, representing rigid circular string with spins J_1, J_2 and two BMN ones with large spins J'_1 and J''_1) that scales as $\exp[-J_1 f(\frac{J_2}{J_1}, \frac{J'_1}{J_1}, \frac{J''_1}{J_1})]$ where f is positive.

¹⁶The case with 3 “heavy” operators is conceptually different from the one considered in [10, 12] and here. If only two operators carry large quantum numbers the leading contribution to the 3-point correlation functions comes from evaluating the “light” vertex operator on the appropriate classical string solution on 2-cylinder. On the other hand, if we consider a correlator with 3 “heavy” operators, the corresponding semiclassical trajectory will no longer be directly related to a known smooth spinning string solution. Instead, it should be describing a semiclassical decay of one large string into two other large strings, with the amplitude proportional to the exponent of the corresponding value of the classical string action, i.e. $\sim e^{-a\sqrt{\lambda}}$ (this factor may cancel out if all 3 states are near-BMN). This exponential contribution will be multiplied also by other factors (not depending on large charges) in the “heavy” vertex operators evaluated on the classical solution. The resulting “pre-exponential” factor will scale at least as $\sqrt{\lambda}$ so will still dominate over quantum string corrections, coming, e.g., from the one-loop fluctuation determinant, which will be of order 1.

$\langle V_{H1}(x_1)V_{H2}(x_2)V_{H3}(x_3)V_{H4}(x_4) \rangle$ may be dominated by the same type of semiclassical contribution as in the case of $\langle V_{H1}V_{H2}V_{L1}V_{L2} \rangle$, i.e. by a sum of products of the semiclassical 3-point functions with intermediate “light” states, i.e. $\langle V_{H1}V_{H2}V_L \rangle \langle V_LV_{H3}V_{H4} \rangle$.

As was already mentioned in [12] and above, a similar semiclassical approach as discussed here on the example of the 4-point functions can be applied also to the study of strong-coupling limit of higher-point correlation functions containing exactly two operators with large charges. The resulting expressions are given by direct generalization of (2.18). It may be of interest to apply them, e.g., to the processes of emission of soft “gravitons” by a semiclassical spinning string. One may also consider a sum of such correlators with the same type of “light” state corresponding to the exponentiation of the “light” vertex operator. This may be of interest for understanding the effect of “back reaction” of the “light” states on the “heavy” one.

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A Structure of chiral primary scalar vertex operator

Let us review the structure of the full chiral primary scalar vertex operator as given in [10] by following [21] and adapting the results of [17, 24]. Assuming that the operator is inserted at point $\vec{x} = 0$ at the boundary of AdS_5 we have ($\mathbb{X}^{\mathbb{M}} = (y^m, X^k)$)

$$V_j(0) = \hat{c}_j \int d^2\sigma \, h_{\mathbb{M}\mathbb{N}}(\mathbb{X}) \, \partial_\alpha \mathbb{X}^{\mathbb{M}} \partial_\alpha \mathbb{X}^{\mathbb{N}} \quad (\text{A.1})$$

$$= \hat{c}_j \int d^2\sigma \, \left[\left(a \, g_{mn}(y) + b \, \mathcal{D}_m \mathcal{D}_n \right) \phi_j(y, X) \, \partial_\alpha y^m \partial_\alpha y^n - \phi_j(y, X) \, \partial_\alpha X_k \partial_\alpha X_k \right]$$

$$\phi_j(y, X) = K_j(y) \, X^j, \quad K_j = \left(\frac{z}{z^2 + \vec{x}^2} \right)^j, \quad X \equiv X_1 + iX_2, \quad (\text{A.2})$$

$$a = \frac{j-1}{j+1}, \quad b = -\frac{2}{j(j+1)}, \quad \hat{c}_j = \frac{\sqrt{\lambda}}{N} \frac{(j+1)\sqrt{j}}{8\pi}. \quad (\text{A.3})$$

Here $y^m = (z, \vec{x})$ with $\vec{x} = \{x^\mu\}$ ($\mu = 0, 1, 2, 3$) are AdS_5 Poincaré patch coordinates with $g_{mn} dy^m dy^n = z^{-2}(dz^2 + dx^\mu dx^\mu)$ and \mathcal{D}_m are the corresponding covariant derivatives. X_k are embedding coordinates of S^5 . ϕ_j solves the free scalar field equation in AdS_5 with mass $M^2 = j(j-4)$ with the boundary condition $\phi_j(z, \vec{x})_{z \rightarrow 0} \sim \delta^{(4)}(\vec{x})$. The above expression for $V_j(0)$ is manifestly $SO(2, 4) \times SO(6)$ covariant.

The covariant derivatives $\mathcal{D}_m \mathcal{D}_n$ act only on the $K_j(y)$ part of ϕ_j and we find

$$\begin{aligned}\mathcal{D}_\mu \mathcal{D}_\nu K_j &= (\partial_\mu \partial_\nu - \frac{1}{z} \delta_{\mu\nu} \partial_z) K_j = \left[-\frac{j}{z^2} \delta_{\mu\nu} + \frac{4j(j+1)}{(z^2 + \vec{x}^2)^2} x_\mu x_\nu \right] K_j, \\ \mathcal{D}_\mu \mathcal{D}_z K_j &= (\partial_\mu \partial_z + \frac{1}{z} \partial_\mu) K_j = \frac{2j(j+1)(z^2 - \vec{x}^2)}{(z^2 + \vec{x}^2)^2} \frac{x_\mu}{z} K_j, \\ \mathcal{D}_z \mathcal{D}_z K_j &= (\partial_z \partial_z + \frac{1}{z} \partial_z) K_j = \left[\frac{j^2}{z^2} - \frac{4j(j+1)\vec{x}^2}{(z^2 + \vec{x}^2)^2} \right] K_j,\end{aligned}\tag{A.4}$$

where we split the index m into the z -component and the boundary μ -components. Substituting (A.4) into (A.1) we end up with the expression quoted in (4.4), (4.6), (4.8), i.e.

$$V_j(0) = \hat{c}_j \int d^2\sigma K_j X^j U, \quad U = U_1 + U_2 + U_3, \tag{A.5}$$

$$U_1 = \frac{1}{z^2} (\partial_\alpha \vec{x} \cdot \partial_\alpha \vec{x} - \partial_\alpha z \partial_\alpha z) - \partial_\alpha X_k \partial_\alpha X_k, \tag{A.6}$$

$$U_2 + U_3 = \frac{8}{(z^2 + \vec{x}^2)^2} \left([\vec{x}^2 (\partial_\alpha z)^2 - (\vec{x} \cdot \partial_\alpha \vec{x})^2] + \frac{1}{z} (\vec{x}^2 - z^2) (\vec{x} \cdot \partial_\alpha \vec{x}) \partial_\alpha z \right). \tag{A.7}$$

Let us now evaluate (A.5) on the solution (4.3) corresponding to the insertion of two “heavy” chiral primary operators. If we choose $a_1 = -a_2$ in (4.3) we find that

$$U_2 = -U_3 = 8\kappa^2 \frac{\tanh^2 \kappa \tau_e}{\cosh^4 \kappa \tau_e} (1 - \sinh^2 \kappa \tau_e), \quad U_2 + U_3 = 0, \tag{A.8}$$

i.e. the result of using full the vertex operator (A.5) in (4.2) is the same as using the “truncated” operator with $U = U_1$ as in [12]. For generic a_1 and a_2 we get

$$\begin{aligned}K_j &= \frac{a_{12}^j}{(a_1^2 e^{\kappa \tau_e} + a_2^2 e^{-\kappa \tau_e})^j}, \quad X^j = e^{j\kappa \tau_e}, \quad U_1 = \frac{2\kappa^2}{\cosh^2(\kappa \tau_e)}, \\ U_2 &= -2\kappa^2 a_{12}^2 \left(\frac{a_1 e^{\kappa \tau_e} + a_2 e^{-\kappa \tau_e}}{a_1^2 e^{\kappa \tau_e} + a_2^2 e^{-\kappa \tau_e}} \right)^2 \frac{1 - \sinh^2 \kappa \tau_e}{\cosh^4 \kappa \tau_e}, \\ U_3 &= 2\kappa^2 a_{12} (a_1 e^{\kappa \tau_e} + a_2 e^{-\kappa \tau_e}) \frac{(a_1 - a_2)^2 - (a_1 e^{\kappa \tau_e} + a_2 e^{-\kappa \tau_e})^2}{(a_1^2 e^{\kappa \tau_e} + a_2^2 e^{-\kappa \tau_e})^2} \frac{\sinh \kappa \tau_e}{\cosh^4 \kappa \tau_e}.\end{aligned}\tag{A.9}$$

This then leads to (4.9), (4.10).

Let us now evaluate the vertex operator (A.5) on the solution (4.25) corresponding to the insertion of two large spin S operators. In this case we find

$$\begin{aligned}K_j &= \frac{a_{12}^j}{\cosh^j \mu \sigma (a_1^2 e^{\kappa \tau_e} + a_2^2 e^{-\kappa \tau_e})^j}, \quad X_1 + iX_2 = 1, \\ U_1 &= \frac{2\kappa^2}{\cosh^2 \kappa \tau_e} - 2\mu^2 \tanh^2 \mu \sigma.\end{aligned}\tag{A.10}$$

The expressions for U_2 and U_3 are rather complicated, but their sum simplifies to

$$U_2 + U_3 = -\frac{2\kappa^2}{\cosh^2 \kappa\tau_e} \frac{(a_1^2 - a_2^2) (a_1^2 e^{2\kappa\tau_e} - a_2^2 e^{-2\kappa\tau_e})}{(a_1^2 e^{\kappa\tau_e} + a_2^2 e^{-\kappa\tau_e})^2}, \quad (\text{A.11})$$

and again vanishes if $a_1 = -a_2$. Substituting (A.10), (A.11) into (A.1) we are led to (4.27).

Let us finish with a remark on an equivalent form of the vertex operator in (A.1) evaluated on y^m, X_k that solve the classical string equations of motion (as in the examples we discussed above). Consider the covariant derivative term in the integrand in (A.1)

$$A \equiv \mathcal{D}_m \mathcal{D}_n (K_j(y) X^j) \partial_\alpha y^m \partial_\alpha y^n = X^j \mathcal{D}_m \mathcal{D}_n K_j \partial_\alpha y^m \partial_\alpha y^n = X^j \nabla_\alpha (\mathcal{D}_n K_j) \partial_\alpha y^n, \quad (\text{A.12})$$

where $\nabla_\alpha \equiv \partial_\alpha y^n \mathcal{D}_n$. Equivalently,

$$A = \nabla_\alpha (X^j \mathcal{D}_n K_j \partial_\alpha y^n) - \mathcal{D}_\alpha X^j \mathcal{D}_n K_j \partial_\alpha y^n - X_j \mathcal{D}_n K_j \nabla_\alpha \partial_\alpha y^n. \quad (\text{A.13})$$

Here the last term vanishes if y^n satisfies the equations of motion (or it can be removed by a 2d field redefinition). Since $\nabla_\alpha (X^j \mathcal{D}_n K_j \partial_\alpha y^n) = \partial_\alpha (X^j \partial_\alpha K_j)$ We are then left with the following equivalent form of (A.1)

$$V_j(0) = \hat{c}_j \int d^2\sigma \left[K_j X^j \left(a g_{mn}(y) \partial_\alpha y^m \partial_\alpha y^n - \partial_\alpha X_k \partial_\alpha X_k \right) + b X^j \partial_\alpha \partial_\alpha K_j \right], \quad (\text{A.14})$$

where the last term can be written also as (dropping total derivative)¹⁷

$$-b \partial_\alpha X^j \partial_\alpha K_j = -j b X^{j-1} \partial_\alpha X \mathcal{D}_n K_j \partial_\alpha y^n. \quad (\text{A.15})$$

These expressions are easy to evaluate on given classical solutions.

B Large J limit of \hat{D} -term in the supergravity expression

Here we shall study the large J limit of the last term in eq. (5.11), i.e.

$$\hat{D}(u, v; J) = \frac{2Ju^J}{(J-2)!} \bar{D}_{J,J+2,2,2}(u, v). \quad (\text{B.1})$$

We shall find that it scales as $J^{1/2}$, i.e. is subleading compared to the second term in (5.11).

¹⁷Note that the presence of this term that “mixes” AdS_5 and S^5 coordinates in the “graviton” vertex operator may be attributed to the mixing with the RR field fluctuations, requiring some graviton field redefinitions.

Using eqs. (5.13), (5.15) and (2.19) we get

$$\hat{D} = \frac{4J(J-1)(J+1)}{\pi^2} \frac{x_{13}^2 x_{24}^4 x_{34}^2}{x_{14}^2} x_{12}^{2J} D_{J,J+2,2,2}. \quad (\text{B.2})$$

Since we are interested only in the leading behavior for large J the J -independent polynomial factors are irrelevant and we can write

$$\hat{D} \sim \frac{2J^3}{\pi^2} x_{12}^{2J} D_{J,J,2,2}, \quad (\text{B.3})$$

where

$$D_{J,J,2,2} = \int \frac{dz d^4x}{z^5} \left[\frac{z}{z^2 + (\vec{x} - \vec{x}_1)^2} \quad \frac{z}{z^2 + (\vec{x} - \vec{x}_2)^2} \right]^J \left[\frac{z}{z^2 + (\vec{x} - \vec{x}_3)^2} \quad \frac{z}{z^2 + (\vec{x} - \vec{x}_4)^2} \right]^2 \quad (\text{B.4})$$

For large J this integral can be evaluated at the stationary point which should solve the equations of motion following from the “action” (we represent the leading term in the integrand as e^{-JA})

$$A = -\ln \frac{z}{z^2 + (\vec{x} - \vec{x}_1)^2} - \ln \frac{z}{z^2 + (\vec{x} - \vec{x}_2)^2}. \quad (\text{B.5})$$

We may assume that all the points \vec{x}_1 and \vec{x}_2 in (B.4) are along the x_{0e} -axis. As before we denote their zeroth components a_1 and a_2 and assume $a_1 > a_2$. Then it is straightforward to show that the solution is

$$z^2 = (a_1 - x_{0e})(x_{0e} - a_2), \quad x_1 = x_2 = x_3 = 0. \quad (\text{B.6})$$

This solution can be parametrized as ($a_{12} = a_1 - a_2$)

$$z = \frac{a_{12}}{2 \cosh \tau}, \quad x_{0e} = \frac{a_{12}}{2} \tanh \tau + \frac{1}{2}(a_1 + a_2), \quad x_1 = x_2 = x_3 = 0. \quad (\text{B.7})$$

Note that this is precisely the point-particle solution in (4.3) (with $\kappa = 1$). Since this stationary point solution is not an isolated point but a line parametrized by τ , the integral over the AdS_5 space in (B.4) should be reduced to the integral over τ . This is done by relating the integration measure to the induced measure on the curve

$$\int \frac{dz d^4x}{z^5} = \int d\tau \sqrt{g_{ind}} = \int d\tau, \quad (\text{B.8})$$

where we used that, as follows from (B.7), the determinant of the induced metric on the curve g_{ind} is equal to 1. On the stationary point solution the “action” in (B.5) is $A = 2 \ln a_{12} = 2 \ln x_{12}$ so that we obtain

$$D_{J,J,2,2} \sim \frac{1}{x_{12}^{2J}} \int_{-\infty}^{\infty} d\tau \left[\frac{z}{z^2 + (\vec{x} - \vec{x}_3)^2} \quad \frac{z}{z^2 + (\vec{x} - \vec{x}_4)^2} \right]^2 Q, \quad (\text{B.9})$$

where Q is the “one-loop” determinant over the fluctuations around the solution (B.7)

$$Q = \int d^5 \delta y \ e^{-\frac{1}{2} J \partial_m \partial_n A \ \delta y^m \delta y^n} \sim J^{-5/2} \det^{-1/2}(\partial_m \partial_n A) , \quad (\text{B.10})$$

where $\delta y^n = (\delta z, \delta \vec{x})$. The integrand in (B.9) has to be evaluated on the solution (B.7). The τ -integral is J -independent, and thus the leading J dependence of (B.9) is given by

$$D_{J,J,2,2} \sim \frac{J^{-5/2}}{x_{12}^{2J}} . \quad (\text{B.11})$$

Substituting this into (B.3) gives

$$\hat{D} \sim J^{1/2} . \quad (\text{B.12})$$

This shows that \hat{D} is subleading compared to (5.16).

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